

# STABLE SCHEDULE MATCHINGS BY A FIXED POINT METHOD

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**ABSTRACT.** We generalize several schedule matching theorems of Baiou–Balinski (Math. Oper. Res., 27 (2002), 485) and Alkan–Gale (J. Econ. Th. 112 (2003), 289) by applying a fixed point method of Fleiner (Math. Oper. Res., 28 (2003), 103). Thanks to a more general construction of revealing choice maps we develop an algorithm to solve rather complex matching problems. The flexibility and efficiency of our approach is illustrated by various examples. We also revisit the mathematical structure of the matching theory by comparing various definitions of stable sets and various classes of choice maps. We demonstrate, by several examples, that the revealing property of the choice maps is the most suitable one to ensure the existence of stable matchings; both from the theoretical and the practical point of view.

## 1. INTRODUCTION

Since the pioneering paper of Gale and Shapley [8] on *stable matchings*, many studies have been devoted to the adaptations and the generalizations of their algorithm. Stable matching algorithms have found use in diverse economic applications ranging from labor markets to college admissions or even kidney exchanges.

In these two-sided matching markets, two sets of agents have preferences over the opposite set: on one side of the market, there are individuals (students, interns or employees) and on the other side there are institutions (colleges, hospitals or firms). A “stable match” is realized when all agents have been matched with the opposite side such that neither could obtain a more mutually beneficial match on their own.

The original strict preference ordering assumptions proved to be too restrictive for many real world problems. Following an influential contribution of Kelso and Crawford [4], Roth [11] made a systematic study of a more flexible approach based on *choice functions*. The monograph of Roth and Sotomayor [13] provides an overview of the state of the art up to 1990 and it still serves as an excellent introduction to the subject. Feder [6], Subramanian [14] and Adachi [1] discovered a close relationship between stable matchings and fixed points of set-valued maps. Then Fleiner [7] demonstrated that many classical results may be obtained by a straightforward application of an old theorem of Knaster [10] and Tarski [15], [16]. See also Hatfield and Milgrom [9] for an economically motivated presentation of the fixed point method.

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*Date:* Version 2010-05-11-a.

Part of this research was realized during the stay of the first author at the Department of Mathematics of the University of Cincinnati as a Taft research fellow in March–June, 2007. He is grateful to the Charles Phelps Taft Research Center for their kind invitation and for the excellent working conditions.

More recently, Baiou and Balinski [3] introduced the notion of *schedule matching* which made it possible to consider, as a part of the contract not only the hiring of a particular worker by a particular firm, but also the number of hours of employment of the worker in the firm. In their setting, “stability” means that no pair of opposite agents can increase their hours together either due to unused capacity or by giving up hours with less desirable partners. They assumed that all agents have strict preference orderings. Alkan and Gale [2] extended their model by using incomplete revealed preference ordering via choice functions instead.

In this paper, we generalize the notion of schedule matching of Baiou and Balinski [3] to allow for schedule and preference constraints on each side of the market. We define a revealing choice map for each agent on the acceptable opposite side agent(s), possible days and (combinations of) restrictions or “subsets” placed on the opposite side agent and/or days worked. In particular, our framework allows for possible quotas placed by workers on firms and days worked, allowing him to work part-time for different firms on the same day or on different days, excluding some firms on some given days or excluding some days of work. In the same manner, it allows firms to adjust their labor force on certain days depending on their anticipated activity, or on the requirements associated to different activities on different days (or the same day). We show that the allocation of days, firms and workers is stable in the sense that given their schedule constraints, their preference orderings and constraints, there is no better schedule for both parties; moreover the stable allocation is shown to be worker optimal or firm optimal. This is done by using a slightly simplified version of Fleiner’s theorem and by giving a general construction of choice maps having the revealed preference property. We illustrate the power of our theorems by several examples. We provide the algorithm that can be used to obtain the optimal allocation: we will solve a deliberately complex example to explain its technical execution. Furthermore, in order to discuss the optimality of our results, we clarify the relationships between various properties of choice maps and between different definitions of stable sets, often used in the literature.

The plan of the paper is the following. In Section 2 we formulate a model problem which will motivate our research and which may have natural real-world applications. In Section 3 we present the mathematical framework for our model. In Section 4 we solve the problems of Section 2 and we also explain how our results cover some of the theorems of Alkan and Gale [2]. In Section 5 we illustrate the power and flexibility of our method by solving a number of more complex problems. Section 6 concludes. The proofs of the theoretical results are given in Section 7.

## 2. SCHEDULE MATCHING PROBLEMS

In order to illustrate the novelty of the present work we begin by recalling the first example of Gale and Shapley [8]. They considered three women:  $w_1$ ,  $w_2$ ,  $w_3$  and three men:  $f_1$ ,  $f_2$ ,  $f_3$  with the following preference orders (we change the notations for consistence with our later examples):

- Preference order of  $w_1$ :  $f_1 \succ f_2 \succ f_3$ ;
- Preference order of  $w_2$ :  $f_2 \succ f_3 \succ f_1$ ;
- Preference order of  $w_3$ :  $f_3 \succ f_1 \succ f_2$ ;
- Preference order of  $f_1$ :  $w_2 \succ w_3 \succ w_1$ ;
- Preference order of  $f_2$ :  $w_3 \succ w_1 \succ w_2$ ;
- Preference order of  $f_3$ :  $w_1 \succ w_2 \succ w_3$ .

They looked for the possibilities of marrying all six people in a stable way. Instability would occur if there were a woman and a man, not married to each other who would prefer each other to their actual mates. It turns out that there are three solutions:

- each woman gets her first choice:  $(w_1, f_1), (w_2, f_2), (w_3, f_3)$ ;
- each man gets his first choice:  $(w_1, f_3), (w_2, f_1), (w_3, f_2)$ ;
- everyone get her or his second choice:  $(w_1, f_2), (w_2, f_3), (w_3, f_1)$ .

Now let us modify the problem to a simple job market problem as follows. Consider three workers:  $w_1, w_2, w_3$  and three firms:  $f_1, f_2, f_3$  with the same preference orders for hiring as above. Furthermore, assume that hiring is for two different days:  $d_1, d_2$ , with the following additional preferences and requirements:

- for each worker–firm pair  $(w_i, f_j)$ , the worker prefers  $d_1$  to  $d_2$  and the firm prefers  $d_2$  to  $d_1$ ;
- each worker may be hired by at most one firm on each given day (maybe different firms on different days);
- each firm may hire at most two workers per day; if they hire two workers for one given day, then they cannot hire anybody for the other day;
- no firm may hire the same worker for both days.

We are looking for a stable set of contracts, i.e., for a set  $S$  of triplets  $(w_i, f_j, d_k)$  having the following properties:

- each contract  $(w_i, f_j, d_k) \in S$  is acceptable for both  $w_i$  and  $f_j$ ;
- for any other contract  $(w_i, f_j, d_k) \notin S$ , either  $w_i$  and/or  $f_j$  prefers her/his contracts in  $S$  to this new one.

The following section presents the theory necessary to address this type of problems. The solution to this example is given in Section 4.

### 3. EXISTENCE OF STABLE SCHEDULE MATCHINGS

In this section we develop the theoretical framework required to solve problems like that of the preceding section. The main results are Theorems 3.7, 3.10 and 3.13. Propositions 3.5 and 3.6 contain useful complements and will be used in the proof of Theorem 3.7 but they are not necessary for the understanding and the applications of our theorems. For the reader's convenience, proofs are postponed to Section 6, which contains various remarks and examples discussing the optimality of the results formulated here.

Given a set  $X$ , we denote by  $2^X$  the set of all subsets of  $X$ . By a *choice map* in  $X$  we mean a function  $C : 2^X \rightarrow 2^X$  satisfying

$$C(A) \subset A \quad \text{for all } A \subset X. \quad (3.1)$$

In economic applications  $X$  is the set of all possible contracts, and for a given set  $A$  of proposed contracts,  $C(A)$  denotes the set of accepted contracts by some given rules of the market.

Assume that there are two competing sides, for example *workers and firms* and correspondingly two choice functions  $C_W, C_F : 2^X \rightarrow 2^X$ .

**Definition 3.1.** A set  $S$  of contracts is said to be *stable* if there exist two sets  $S_W, S_F \subset X$  satisfying the following three conditions:

$$S_W \cup S_F = X; \quad (3.2)$$

$$C_W(A) = S \quad \text{for every } S \subset A \subset S_W; \quad (3.3)$$

$$C_F(A) = S \quad \text{for every } S \subset A \subset S_F. \quad (3.4)$$

Stable contract sets represent acceptable compromises.

*Remark 3.2.* A stable set<sup>1</sup>  $S$  is *individually rational* if

$$C_W(S) = S = C_F(S), \quad (3.5)$$

and it is *not blocked by any other contract*, i.e., for each  $x \in X$  we have

$$\text{either } C_W(S \cup \{x\}) = S \quad \text{or} \quad C_F(S \cup \{x\}) = S \quad (\text{or both}). \quad (3.6)$$

In order to ensure the existence of stable sets of contracts we need one additional assumption on the choice maps.

**Definition 3.3.** We say that a choice map  $C : 2^X \rightarrow 2^X$  is *revealing* (or satisfies the *revealed preference condition*) if

$$A, B \subset X \quad \text{and} \quad C(A) \subset B \implies A \cap C(B) \subset C(A). \quad (3.7)$$

This means that if a contract is rejected from some proposed set  $A$  of contracts, then it will also be rejected from every other proposed set  $B$  which contains the accepted contracts.

*Example 3.4.*

(a) For any fixed set  $Y \subset X$  the formula  $C(A) := A \cap Y$  defines a revealing choice map on  $X$ . This example illustrates a situation where some contracts are unacceptable to certain agents.

(b) More generally, given a finite subset  $Y \subset X$ , a nonnegative integer  $q$  (called *quota*) and a strict preference ordering  $y_1 \succ y_2 \succ \dots$  on  $Y$ , we define a map  $C(A)$  for any given  $A \subset X$  as follows. If  $\text{Card}(A \cap Y) \leq q$ , then we set  $C(A) := A \cap Y$ . If  $\text{Card}(A \cap Y) > q$ , then let  $C(A)$  be the set of the first  $q$  elements of  $A \cap Y$  according to the ordering of  $Y$ . Then  $C : 2^X \rightarrow 2^X$  is a revealing choice map on  $X$ .

Choice maps of this kind are frequently used in classical matching problems such as the marriage problem, the college admission problem and various many-to-many matching problems; see, e.g., [2], [13] and the references of the latter.

Before stating our main theorem, we further clarify the relationships between the revealed preference condition and other usual properties of choice maps (Proposition 3.5.). We also discuss alternative equivalent definitions of stable sets (Proposition 3.6.).

**Proposition 3.5.**

(a) A choice map  $C : 2^X \rightarrow 2^X$  is revealing if and only if it is consistent:

$$C(A) \subset B \subset A \implies C(B) = C(A) \quad (3.8)$$

and persistent (or satisfies the substitute condition):

$$A \subset B \implies A \cap C(B) \subset C(A). \quad (3.9)$$

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<sup>1</sup>A more thorough investigation of stable sets is carried out in Proposition 3.6 below.

(b) A choice map is persistent if and only if the rejection map  $R : 2^X \rightarrow 2^X$  defined by  $R(A) := A \setminus C(A)$  is monotone, i.e.,

$$A \subset B \implies R(A) \subset R(B). \quad (3.10)$$

(c) A choice map satisfying either (3.8) or (3.9) is idempotent:

$$C(C(A)) = C(A) \quad \text{for all } A \subset X. \quad (3.11)$$

**Proposition 3.6.** We consider two choice maps  $C_W, C_F : 2^X \rightarrow 2^X$  and three sets  $S, S_W, S_F \subset X$  satisfying

$$S_W \cup S_F = X \quad \text{and} \quad C_W(S_W) = S = C_F(S_F). \quad (3.12)$$

(a) If the choice maps  $C_W, C_F : 2^X \rightarrow 2^X$  are idempotent, then  $S$  is individually rational, i.e., it satisfies (3.5).

(b) If at least one of the two choice maps  $C_W, C_F : 2^X \rightarrow 2^X$  is consistent, then we may modify  $S_W$  or  $S_F$  such that  $S_W \cap S_F = S$  and (3.12) remains valid.

(c) If, moreover, both choice maps are consistent, then (3.12) is equivalent to the stability (3.2)–(3.4) of  $S$ .

(d) If both choice maps  $C_W, C_F : 2^X \rightarrow 2^X$  are revealing, then a set  $S$  is stable if and only if it is individually rational, and it is not blocked by any other contract, i.e., (3.2)–(3.4) are equivalent to (3.5)–(3.6) (for all  $x \in X$ ).

Observe that property (3.12) below follows from the definition of stable sets.

Our main theorem below shows that the revealed preference condition ensures the existence of stable sets of contracts:

**Theorem 3.7.** If  $C_W, C_F : 2^X \rightarrow 2^X$  are two revealing choice maps, then there exists at least one stable set of contracts.

*Remark 3.8.*

(a) The proof of the theorem, provided in Section 7, will show that the stable sets form a complete lattice for a natural order relation. In particular, there exists a worker-optimal and a firm-optimal stable set.

(b) In case  $X$  is a finite set, the proof of the theorem provides an efficient algorithm to find a stable set. Starting with  $X_0 := X$  we compute successively  $Y_1, X_2, Y_3, X_4, \dots$  by using the recursive formulae

$$Y_{n+1} := (X \setminus X_n) \cup C_W(X_n) \quad \text{and} \quad X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n).$$

There exists a first index  $n \geq 1$  such that  $X_{n-1} = X_{n+1}$ , and then  $S := C_W(X_{n-1})$  is the worker-optimal stable set.

Similarly, starting with  $Y_0 := X$  we may compute successively  $X_1, Y_2, X_3, Y_4, \dots$  by the same recursive formulae. There exists a first index  $n \geq 1$  such that  $Y_{n-1} = Y_{n+1}$ , and then  $S := C_F(Y_{n-1})$  is the firm-optimal stable set. See Remark 7.2 below for the details.

(c) The definitions of revealing choice maps, stable sets, the theorem and the preceding remarks remain valid if we replace  $2^X$  by a complete sublattice  $L$  of  $2^X$ , i.e., a subfamily  $L$  of  $2^X$  such that the union and the intersection of any system of sets  $A \in L$  still belongs to  $L$ . See, e.g., [7] for more details on lattice properties.

(d) Part (a) of Proposition 3.5 shows that Theorem 3.7 is mathematically equivalent to a theorem of Fleiner [7].

(e) We will show in Examples 7.1 (a)–(b) and 7.5 of Section 6 that the revealing condition cannot be weakened in Theorem 3.7.

In order to apply Theorem 3.7 for the solution of the problem stated in Section 2, we need a generalization of the construction of revealing choice maps recalled in Example 3.4. Such a construction is provided by Theorem 3.10 below.

Let us be given a finite subset  $Y \subset X$ , a family  $\{Y_n\}$  of subsets  $Y_n \subset X$ , and corresponding nonnegative integers (called *quotas*)  $q$  and  $q_n$ . We assume that the sets  $Y_n \cap Y$  are disjoint. Furthermore, let us be given a strict preference ordering  $y_1 \succ y_2 \succ \dots$  on  $Y$ . Given any set  $A \subset X$ , we define a nondecreasing sequence  $C_0(A) \subset C_1(A) \subset \dots$  of subsets of  $A \cap Y$  by recursion as follows. First we set  $C_0(A) = \emptyset$ . If  $C_{k-1}(A)$  has already been defined for some  $k$ , then we set  $C_k(A) := C_{k-1}(A) \cup \{y_k\}$  if

$$\begin{aligned} y_k &\in A, \\ \text{Card } C_{k-1}(A) &< q, \\ \text{Card } (C_{k-1}(A) \cap Y_n) &< q_n \text{ if } y_k \in Y_n; \end{aligned}$$

otherwise we set  $C_k(A) := C_{k-1}(A)$ . Finally, we define  $C(A) := \cup C_k(A)$ .

*Remark 3.9.* It follows from the construction that

$$C(A) \subset A \cap Y; \quad (3.13)$$

$$\text{Card } C(A) \leq q; \quad (3.14)$$

$$\text{Card } (C(A) \cap Y_n) \leq q_n \text{ for all } n. \quad (3.15)$$

**Theorem 3.10.**  $C : 2^X \rightarrow 2^X$  is a revealing choice map.

*Remark 3.11.*

- (a) If  $q_n \geq q$  or  $q_n \geq \text{Card}(Y)$  for some  $n$ , then we may eliminate  $Y_n$  and  $q_n$  without changing the construction.
- (b) If there are no sets  $Y_n$ , then our construction reduces to Example 3.4 (b).
- (c) If, moreover,  $q \geq \text{Card}(Y)$ , then our construction reduces to Example 3.4 (a). (In this case the choice of the order relation is irrelevant.)
- (d) Instead of a finite subset  $Y \subset X$ , we can also consider arbitrary subsets  $Y \subset X$  with a well-ordered preference relation: the construction and the proof of the proposition remain valid.

*Example 3.12.* The disjointness condition is necessary. To show this, consider the sets  $X = Y = \{a, b, c\}$ ,  $Y_1 = \{a, b\}$ ,  $Y_2 = \{b, c\}$  with the quotas  $q = 2$ ,  $q_1 = q_2 = 1$  and the preference order  $a \succ b \succ c$ . Then for  $A = \{b, c\}$  and  $B = \{a, b, c\}$  we have  $C(A) = \{b\}$  and  $C(B) = \{a, c\}$ , so that  $A \subset B$  but  $A \cap C(B) \not\subset C(A)$ .

Theorem 3.10 can be often used for the construction of *individual* revealing choice functions. The following result enables us to combine individual revealing choice functions into *global* revealing choice functions.

**Theorem 3.13.** Given a set function  $C : 2^X \rightarrow 2^X$  and a partition  $X = \cup X_i$  with disjoint sets  $X_i$ , we define the set functions  $C_i : 2^{X_i} \rightarrow 2^{X_i}$  by the formula  $C_i(A_i) := C(A_i) \cap X_i$ . Then  $C$  is a revealing choice map on  $X$  if and only if each  $C_i$  is a revealing choice map on  $X_i$ .

## 4. SOLUTION TO THE SIMPLE JOB MARKET PROBLEM

For the solution we set

$$W := \{w_1, w_2, w_3\}, \quad F := \{f_1, f_2, f_3\}, \quad D := \{d_1, d_2\}$$

and we proceed in several steps.

*Step 1.* For each fixed worker  $w_i$  we define a revealing choice map  $C_{w_i}$  on  $\{w_i\} \times F \times D$  by applying Theorem 3.10 with  $Y$ ,  $q$ ,  $Y_n$  and  $q_n$  given below. For brevity we write  $(i, j, k)$  instead of  $(w_i, f_j, d_k)$  in the preference relations.

- For worker  $w_1$  we choose

$$\begin{aligned} Y &:= \{w_1\} \times \{f_1, f_2, f_3\} \times \{d_1, d_2\}, \\ Y_1 &:= \{w_1\} \times \{f_1, f_2, f_3\} \times \{d_1\}, \\ Y_2 &:= \{w_1\} \times \{f_1, f_2, f_3\} \times \{d_2\} \end{aligned}$$

with quotas  $q = 6$  (which is ineffective),  $q_1 = q_2 = 1$  and the following preference relation on  $Y$ :

$$(1, 1, 1) \succ (1, 1, 2) \succ (1, 2, 1) \succ (1, 2, 2) \succ (1, 3, 1) \succ (1, 3, 2).$$

- For worker  $w_2$  we choose

$$\begin{aligned} Y &:= \{w_2\} \times \{f_1, f_2, f_3\} \times \{d_1, d_2\}, \\ Y_1 &:= \{w_2\} \times \{f_1, f_2, f_3\} \times \{d_1\}, \\ Y_2 &:= \{w_2\} \times \{f_1, f_2, f_3\} \times \{d_2\} \end{aligned}$$

with quotas  $q = 6$ ,  $q_1 = q_2 = 1$  and the following preference relation on  $Y$ :

$$(2, 2, 1) \succ (2, 2, 2) \succ (2, 3, 1) \succ (2, 3, 2) \succ (2, 1, 1) \succ (2, 1, 2).$$

- For worker  $w_3$  we choose

$$\begin{aligned} Y &:= \{w_3\} \times \{f_1, f_2, f_3\} \times \{d_1, d_2\}, \\ Y_1 &:= \{w_3\} \times \{f_1, f_2, f_3\} \times \{d_1\}, \\ Y_2 &:= \{w_3\} \times \{f_1, f_2, f_3\} \times \{d_2\} \end{aligned}$$

with quotas  $q = 6$ ,  $q_1 = q_2 = 1$  and the following preference relation on  $Y$ :

$$(3, 3, 1) \succ (3, 3, 2) \succ (3, 1, 1) \succ (3, 1, 2) \succ (3, 2, 1) \succ (3, 2, 2).$$

*Step 2.* Applying Theorem 3.13 we combine the three choice maps of the preceding step into a global revealing choice map  $C_W$  on  $W \times F \times D$  by setting

$$C_W(A) := \bigcup_{i=1}^3 C_{w_i}(A \cap (\{w_i\} \times F \times D))$$

for every  $A \subset W \times F \times D$ .

*Step 3.* For each firm  $f_j$  we define a revealing choice map  $C_{f_j}$  on  $W \times \{f_j\} \times D$  by applying Theorem 3.10 with  $Y$ ,  $q$ ,  $Y_n$  and  $q_n$  given below and still writing  $(i, j, k)$  instead of  $(w_i, f_j, d_k)$  for brevity.

- For firm  $f_1$  we choose

$$\begin{aligned} Y &:= \{w_1, w_2, w_3\} \times \{f_1\} \times \{d_1, d_2\}, \\ Y_1 &:= \{w_1\} \times \{f_1\} \times \{d_1, d_2\}, \\ Y_2 &:= \{w_2\} \times \{f_1\} \times \{d_1, d_2\}, \\ Y_3 &:= \{w_3\} \times \{f_1\} \times \{d_1, d_2\}, \end{aligned}$$

with quotas  $q = 2$ ,  $q_1 = q_2 = q_3 = 1$  and the following preference relation on  $Y$ :

$$(2, 1, 2) \succ (2, 1, 1) \succ (3, 2, 2) \succ (3, 2, 1) \succ (1, 3, 2) \succ (1, 3, 1).$$

- For firm  $f_2$  we choose

$$\begin{aligned} Y &:= \{w_1, w_2, w_3\} \times \{f_2\} \times \{d_1, d_2\}, \\ Y_1 &:= \{w_1\} \times \{f_2\} \times \{d_1, d_2\}, \\ Y_2 &:= \{w_2\} \times \{f_2\} \times \{d_1, d_2\}, \\ Y_3 &:= \{w_3\} \times \{f_2\} \times \{d_1, d_2\}, \end{aligned}$$

with quotas  $q = 2$ ,  $q_1 = q_2 = q_3 = 1$  and the following preference relation on  $Y$ :

$$(3, 2, 2) \succ (3, 2, 1) \succ (1, 2, 2) \succ (1, 2, 1) \succ (2, 2, 2) \succ (2, 2, 1).$$

- For firm  $f_3$  we choose

$$\begin{aligned} Y &:= \{w_1, w_2, w_3\} \times \{f_3\} \times \{d_1, d_2\}, \\ Y_1 &:= \{w_1\} \times \{f_3\} \times \{d_1, d_2\}, \\ Y_2 &:= \{w_2\} \times \{f_3\} \times \{d_1, d_2\}, \\ Y_3 &:= \{w_3\} \times \{f_3\} \times \{d_1, d_2\}, \end{aligned}$$

with quotas  $q = 2$ ,  $q_1 = q_2 = q_3 = 1$  and the following preference relation on  $Y$ :

$$(1, 3, 2) \succ (1, 3, 1) \succ (2, 3, 2) \succ (2, 3, 1) \succ (3, 3, 2) \succ (3, 3, 1).$$

*Step 4.* Applying Theorem 3.13 we combine the three choice maps of the preceding step into a global revealing choice map  $C_F$  on  $W \times F \times D$  by setting

$$C_F(A) := \bigcup_{j=1}^3 C_{f_j}(A \cap (W \times \{f_j\} \times D)), \quad A \subset W \times F \times D.$$

*Step 5.* The choice maps  $C_W$  and  $C_F$  satisfy the hypotheses of Theorem 3.7. We apply the algorithm as described in Remark 3.8 (b) by starting with  $X_0 := X$  and computing  $Y_1, X_2, Y_3, X_4$  by the formulae

$$Y_{n+1} := (X \setminus X_n) \cup C_W(X_n) \quad \text{and} \quad X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n).$$

We obtain that  $X_2 = X_4$  and therefore  $S = C_W(X_2)$ . The results are summarized in the following table:



$w_i$	$f_j$	$d_k$	$X_0$	$Y_1$	$X_2$	$Y_3$	$X_4$	$S$
1	1	1	$x$	$x$		$x$		
1	1	2	$x$	$x$	$x$	$x$	$x$	$x$
1	2	1	$x$		$x$	$x$	$x$	$x$
1	2	2	$x$		$x$		$x$	
1	3	1	$x$		$x$		$x$	
1	3	2	$x$		$x$		$x$	
2	1	1	$x$		$x$		$x$	
2	1	2	$x$		$x$		$x$	
2	2	1	$x$	$x$		$x$		
2	2	2	$x$	$x$	$x$	$x$	$x$	$x$
2	3	1	$x$		$x$	$x$	$x$	$x$
2	3	2	$x$		$x$		$x$	
3	1	1	$x$		$x$	$x$	$x$	$x$
3	1	2	$x$		$x$		$x$	
3	2	1	$x$		$x$		$x$	
3	2	2	$x$		$x$		$x$	
3	3	1	$x$	$x$		$x$		
3	3	2	$x$	$x$	$x$	$x$	$x$	$x$

In this *worker-optimal* solution each worker is hired by the second most preferred firm for the first day and by the most preferred firm for the second day.

*Step 6.* Applying the algorithm of Remark 3.8 (b) by starting with  $Y_0 := X$  and computing  $X_1, Y_2, X_3, Y_4$  by the above formulae we obtain that  $Y_2 = Y_4$  and therefore  $S = C_F(Y_2)$ . The results are summarized in the following table:

$w_i$	$f_j$	$d_k$	$Y_0$	$X_1$	$Y_2$	$X_3$	$Y_4$	$S$
1	1	1	$x$		$x$		$x$	
1	1	2	$x$		$x$		$x$	
1	2	1	$x$		$x$		$x$	
1	2	2	$x$	$x$	$x$	$x$	$x$	$x$
1	3	1	$x$		$x$	$x$	$x$	$x$
1	3	2	$x$	$x$		$x$		
2	1	1	$x$		$x$	$x$	$x$	$x$
2	1	2	$x$	$x$		$x$		
2	2	1	$x$		$x$		$x$	
2	2	2	$x$		$x$		$x$	
2	3	1	$x$		$x$		$x$	
2	3	2	$x$	$x$	$x$	$x$	$x$	$x$
3	1	1	$x$		$x$		$x$	
3	1	2	$x$	$x$	$x$	$x$	$x$	$x$
3	2	1	$x$		$x$	$x$	$x$	$x$
3	2	2	$x$	$x$		$x$		
3	3	1	$x$		$x$		$x$	
3	3	2	$x$		$x$		$x$	

In this *firm-optimal* solution each firm hires the most preferred worker for the first day and by the second most preferred worker for the second day.

*Remark 4.1.* The stable schedule matchings as studied by Baiou and Balinski [3] and Alkan and Gale [2] enter the present framework as a special case. For simplicity

we consider the discrete case and we denote by  $D = \{1, 2, \dots\}$  the possible number of working hours with  $k$  meaning the  $k$ th working hour. For each worker  $w_i$ , if there is a preference ranking  $f_{j_1} \succ f_{j_2} \succ \dots$  among the firms, then we extend it to the preference ranking

$$\begin{aligned} & (i, j_1, 1) \succ (i, j_1, 2) \succ \dots \succ (i, j_1, q_{i,j_1}^w) \\ & \succ (i, j_2, 1) \succ (i, j_2, 2) \succ \dots \succ (i, j_2, q_{i,j_2}^w) \\ & \succ \dots \\ & \vdots \end{aligned}$$

where  $q_{i,j}^w$  denotes the maximum number of working hours accepted by worker  $w_i$  in firm  $f_j$ . Similarly, for each firm  $f_j$ , if there is a preference ranking  $w_{i_1} \succ w_{i_2} \succ \dots$  among the workers, then we extend it to the preference ranking

$$\begin{aligned} & (i_1, j, 1) \succ (i_1, j, 2) \succ \dots \succ (i_1, j, q_{i_1,j}^f) \\ & \succ (i_2, j, 1) \succ (i_2, j, 2) \succ \dots \succ (i_2, j, q_{i_2,j}^f) \\ & \succ \dots \\ & \vdots \end{aligned}$$

where  $q_{i,j}^f$  denotes the maximum number of working hours accepted by firm  $f_j$  for worker  $w_i$ . Once a stable set  $S$  found, the number of working hours of worker  $w_i$  in firm  $f_j$  is the biggest integer  $k$  such that  $(i, j, k) \in S$ .

## 5. MORE COMPLEX EXAMPLES

We illustrate in this section the strength and flexibility of our theorems and algorithms by solving some more complex problems.

We consider the following modeling issue. We are given a finite number of workers  $w_i$ , firms  $f_j$  and days  $d_k$  (days of a week or days of a month for instance). Each worker may work at one or several firms per day, maybe at different firms on different days. Similarly, each firm may hire a given number of workers per day, maybe different numbers on different days.

A *contract* is by definition a triple  $(w_i, f_j, d_k)$  meaning that worker  $w_i$  is hired by firm  $f_j$  for day  $d_k$ , and we are looking for an acceptable set of contracts, subject to various requirements of both workers and firms. Thus, each worker  $w_i$

- may exclude some firm–day pairs  $(f_j, d_k)$  considered unacceptable;
- has a strict preference ordering among the remaining firm–day pairs;
- may put some other restrictions, such as
  - to set a maximum quota  $q_i^w$  of accepted firm–day pairs;
  - not to work on day  $d_k$  at more than a given number  $q_{i,k}^w$  of firms;
  - or not to work at firm  $f_j$  more than a given number  $\bar{q}_{i,j}^w$  of days.

Similarly, each firm  $f_j$

- may exclude some worker–day pairs  $(w_i, d_k)$  considered unacceptable;
- has a strict preference ordering among the remaining worker–day pairs;
- may put some other restrictions, such as
  - to set a maximum quota  $q_j^f$  of worker–day pairs for hiring;
  - not to hire on day  $d_k$  more than a given number  $q_{j,k}^f$  of workers;

- or not to hire worker  $w_i$  for more than a given number  $\tilde{q}_{i,j}^f$  of days.

*Remarks 5.1.*

- (a) Although we keep strict preference ordering on worker-day pairs or on firm-day pairs, these preference ordering are not sufficient to characterize the choice map of each agent: they also depend on the quota system.
- (b) In most applications we may assume that a worker does not work at more than one firm per day, so that  $q_{i,k}^w = 1$  for every  $k$ ; then  $q_i$  means the maximum number of working days for the worker  $w_i$ .

**5.1. First problem.** Assume that we have four workers  $w_1, w_2, w_3, w_4$  and three firms  $f_1, f_2, f_3$ . Each worker may work at most at one firm per day, maybe at different firms on different days of the week. The further requirements of the agents are listed below.

- Worker  $w_1$  can work at most 4 days per week, with the following strict preference order of the firm–day pairs  $(f_j, d_k)$  where we write  $(j, k)$  instead of  $(f_j, d_k)$  for brevity:

$$\begin{aligned} (2, 1) \succ (3, 1) \succ (2, 2) \succ (3, 2) \succ (2, 3) \succ (3, 3) \succ (2, 4) \\ \succ (3, 4) \succ (2, 5) \succ (3, 5) \succ (2, 6) \succ (3, 6) \succ (2, 7) \succ (3, 7). \end{aligned} \quad (5.1)$$

This list shows for instance that worker  $w_1$  prefers most to be hired by firm  $f_2$  for Mondays ( $d_1$ ), then by firm  $f_3$  always for Mondays, next by firm  $f_2$  for Tuesdays ( $d_2$ ), and so on. The absence of firm  $f_1$  in the list shows that worker  $w_1$  refuses to be hired by that firm.

- Worker  $w_2$  can work at most 3 days per week, with the following strict preference order:

$$\begin{aligned} (1, 1) \succ (1, 2) \succ (1, 3) \succ (1, 4) \succ (1, 5) \succ (1, 6) \succ (1, 7) \\ \succ (2, 1) \succ (2, 2) \succ (2, 3) \succ (2, 4) \succ (2, 5) \succ (2, 6) \succ (2, 7) \\ \succ (3, 1) \succ (3, 2) \succ (3, 3) \succ (3, 4) \succ (3, 5) \succ (3, 6) \succ (3, 7). \end{aligned} \quad (5.2)$$

- Worker  $w_3$  can work at most 2 days per week, with the following strict preference order:

$$\begin{aligned} (2, 2) \succ (2, 3) \succ (3, 2) \succ (3, 3) \succ (1, 2) \succ (1, 3) \\ \succ (2, 4) \succ (2, 5) \succ (2, 6) \succ (1, 4) \succ (1, 5) \succ (1, 6) \\ \succ (3, 6) \succ (3, 4) \succ (3, 5). \end{aligned} \quad (5.3)$$

The list shows in particular that he/she does not work on Mondays ( $d_1$ ) and Sundays ( $d_7$ ).

- Worker  $w_4$  accepts to work on all days of the week, with the following strict preference order:

$$\begin{aligned} (1, 1) \succ (1, 2) \succ (1, 3) \succ (1, 4) \succ (1, 5) \succ (1, 6) \succ (1, 7) \\ \succ (2, 1) \succ (2, 2) \succ (2, 3) \succ (2, 4) \succ (2, 5) \succ (2, 6) \succ (2, 7) \\ \succ (3, 1) \succ (3, 2) \succ (3, 3) \succ (3, 4) \succ (3, 5) \succ (3, 6) \succ (3, 7). \end{aligned} \quad (5.4)$$

- Firm  $f_1$  may hire up to 4 workers per day when it is open, with the following strict preference order of the worker–day pairs  $(w_i, d_k)$  where we write  $(i, k)$

instead of  $(w_i, d_k)$  for brevity:

$$\begin{aligned} (1, 1) &\succ (1, 2) \succ (1, 3) \succ (1, 4) \succ (1, 5) \succ (1, 6) \\ &\succ (2, 1) \succ (2, 2) \succ (2, 3) \succ (2, 4) \succ (2, 5) \succ (2, 6) \\ &\succ (3, 1) \succ (3, 2) \succ (3, 3) \succ (3, 4) \succ (3, 5) \succ (3, 6). \end{aligned} \quad (5.5)$$

The list shows in particular that the firm is closed on Sundays and that it doesn't hire worker  $w_4$ . Otherwise, it prefers most to hire worker  $w_1$  for Mondays, then worker  $w_1$  for Tuesdays, and so on.

- Firm  $f_2$  may also hire up to 4 workers per day, with the following strict preference order:

$$\begin{aligned} (3, 7) &\succ (3, 6) \succ (3, 5) \succ (3, 4) \succ (3, 3) \succ (3, 2) \succ (3, 1) \\ &\succ (4, 7) \succ (4, 6) \succ (4, 5) \succ (4, 4) \succ (4, 3) \succ (4, 2) \succ (4, 1) \\ &\succ (1, 7) \succ (1, 6) \succ (1, 5) \succ (1, 4) \succ (1, 3) \succ (1, 2) \succ (1, 1) \\ &\succ (2, 7) \succ (2, 6) \succ (2, 5) \succ (2, 4) \succ (2, 3) \succ (2, 2) \succ (2, 1). \end{aligned} \quad (5.6)$$

- Firm  $f_3$  is closed on Saturdays and Sundays; for the other days it may hire up to 4 workers per day, with the following strict preference order:

$$\begin{aligned} (4, 1) &\succ (4, 2) \succ (4, 3) \succ (4, 4) \succ (4, 5) \\ &\succ (3, 1) \succ (3, 2) \succ (3, 3) \succ (3, 4) \succ (3, 5) \\ &\succ (2, 1) \succ (2, 2) \succ (2, 3) \succ (2, 4) \succ (2, 5) \\ &\succ (1, 1) \succ (1, 2) \succ (1, 3) \succ (1, 4) \succ (1, 5). \end{aligned} \quad (5.7)$$

Our task is to find an acceptable firm-worker assignment and work schedule under these constraints. For the solution we set

$$W := \{w_1, w_2, w_3, w_4\}, \quad F := \{f_1, f_2, f_3\}, \quad D := \{d_1, \dots, d_7\}$$

and we proceed in several steps.

*Step 1.* For each fixed worker  $w_i$  we define a revealing choice map  $C_{w_i}$  on  $\{w_i\} \times F \times D$  by applying Theorem 3.10 with  $Y := Y_i^w$ ,  $q := q_i^w$ ,  $Y_n := Y_{i,n}^w$  and  $q_n := q_{i,n}^w$  given below. For brevity we write  $(i, j, k)$  instead of  $(w_i, f_j, d_k)$  in the preference relations.

- For worker  $w_1$  we choose

$$Y_1^w := \{w_1\} \times \{f_2, f_3\} \times D$$

representing the set of acceptable firms and days of worker  $w_1$ , with quota  $q_1^w = 4$  and the following preference relation on  $Y_1^w$  (see (5.1)):

$$\begin{aligned} (1, 2, 1) &\succ (1, 3, 1) \succ (1, 2, 2) \succ (1, 3, 2) \succ (1, 2, 3) \\ &\succ (1, 3, 3) \succ (1, 2, 4) \succ (1, 3, 4) \succ (1, 2, 5) \succ (1, 3, 5) \\ &\succ (1, 2, 6) \succ (1, 3, 6) \succ (1, 2, 7) \succ (1, 3, 7). \end{aligned}$$

Furthermore, we set

$$Y_{1,k}^w := \{w_1\} \times F \times \{d_k\} \text{ and } q_{1,k}^w = 1 \text{ for } k = 1, \dots, 7.$$

- For worker  $w_2$  we choose

$$Y_2^w := \{w_2\} \times F \times D$$

with quota  $q_2^w = 3$  and the preference relation

$$\begin{aligned} & (2, 1, 1) \succ (2, 1, 2) \succ (2, 1, 3) \succ (2, 1, 4) \succ (2, 1, 5) \\ & \succ (2, 1, 6) \succ (2, 1, 7) \succ (2, 2, 1) \succ (2, 2, 2) \succ (2, 2, 3) \\ & \succ (2, 2, 4) \succ (2, 2, 5) \succ (2, 2, 6) \succ (2, 2, 7) \succ (2, 3, 1) \\ & \succ (2, 3, 2) \succ (2, 3, 3) \succ (2, 3, 4) \succ (2, 3, 5) \succ (2, 3, 6) \succ (2, 3, 7). \end{aligned}$$

on  $Y_2^w$  (see (5.2)). Furthermore, we set

$$Y_{2,k}^w := \{w_2\} \times F \times \{d_k\} \text{ and } q_{2,k}^w = 1 \text{ for } k = 1, \dots, 7.$$

- For worker  $w_3$  we choose

$$Y_3^w := \{w_3\} \times F \times \{d_2, \dots, d_6\}$$

with quota  $q_3^w = 2$  and the preference relation

$$\begin{aligned} & (3, 2, 2) \succ (3, 2, 3) \succ (3, 3, 2) \succ (3, 3, 3) \succ (3, 1, 2) \\ & \succ (3, 1, 3) \succ (3, 2, 4) \succ (3, 2, 5) \succ (3, 2, 6) \succ (3, 1, 4) \\ & \succ (3, 1, 5) \succ (3, 1, 6) \succ (3, 3, 6) \succ (3, 3, 4) \succ (3, 3, 5). \end{aligned}$$

on  $Y_3^w$  (see (5.3)). Furthermore, we set

$$Y_{3,k}^w := \{w_3\} \times F \times \{d_k\} \text{ and } q_{3,k}^w = 1 \text{ for } k = 1, \dots, 7.$$

- For worker  $w_4$  we choose

$$Y_4^w := \{w_4\} \times F \times D$$

with quota  $q_4^w = 7$  and the preference relation

$$\begin{aligned} & (4, 1, 1) \succ (4, 1, 2) \succ (4, 1, 3) \succ (4, 1, 4) \succ (4, 1, 5) \\ & \succ (4, 1, 6) \succ (4, 1, 7) \succ (4, 2, 1) \succ (4, 2, 2) \succ (4, 2, 3) \\ & \succ (4, 2, 4) \succ (4, 2, 5) \succ (4, 2, 6) \succ (4, 2, 7) \succ (4, 3, 1) \\ & \succ (4, 3, 2) \succ (4, 3, 3) \succ (4, 3, 4) \succ (4, 3, 5) \succ (4, 3, 6) \succ (4, 3, 7). \end{aligned}$$

on  $Y_4^w$  (see (5.4)). Furthermore, we set

$$Y_{4,k}^w := \{w_4\} \times F \times \{d_k\} \text{ and } q_{4,k}^w = 1 \text{ for } k = 1, \dots, 7.$$

*Step 2.* Applying Theorem 3.13 we combine the four choice maps of the preceding step into a global revealing choice map  $C_W$  on  $W \times F \times D$  by setting

$$C_W(A) := \bigcup_{i=1}^4 C_{w_i}(A \cap (\{w_i\} \times F \times D))$$

for every  $A \subset W \times F \times D$ .

*Step 3.* For each firm  $f_j$  we define a revealing choice map  $C_{f_j}$  on  $W \times \{f_j\} \times D$  by applying Theorem 3.10 again, this time with  $Y := Y_j^f$ ,  $q := q_j^f$ ,  $Y_n := Y_{j,n}^f$  and  $q_n := q_{j,n}^f$  given below and still writing  $(i, j, k)$  instead of  $(w_i, f_j, d_k)$  for brevity.

- For firm  $f_1$  we choose

$$Y_1^f := W \times \{f_1\} \times \{d_1, \dots, d_6\}$$

with quota  $q_1^f = 24$  and the following preference relation on  $Y_1^f$  (see (5.5)):

$$\begin{aligned} (1, 1, 1) &\succ (1, 1, 2) \succ (1, 1, 3) \succ (1, 1, 4) \succ (1, 1, 5) \\ &\succ (1, 1, 6) \succ (2, 1, 1) \succ (2, 1, 2) \succ (2, 1, 3) \succ (2, 1, 4) \\ &\succ (2, 1, 5) \succ (2, 1, 6) \succ (3, 1, 1) \succ (3, 1, 2) \succ (3, 1, 3) \\ &\succ (3, 1, 4) \succ (3, 1, 5) \succ (3, 1, 6). \end{aligned}$$

- For firm  $f_2$  we choose

$$Y_2^f := W \times \{f_2\} \times D$$

with quota  $q_2^f = 28$  and the following preference relation on  $Y_2^f$  (see (5.7)):

$$\begin{aligned} (3, 2, 7) &\succ (3, 2, 6) \succ (3, 2, 5) \succ (3, 2, 4) \succ (3, 2, 3) \\ &\succ (3, 2, 2) \succ (3, 2, 1) \succ (4, 2, 7) \succ (4, 2, 6) \succ (4, 2, 5) \\ &\succ (4, 2, 4) \succ (4, 2, 3) \succ (4, 2, 2) \succ (4, 2, 1) \succ (1, 2, 7) \\ &\succ (1, 2, 6) \succ (1, 2, 5) \succ (1, 2, 4) \succ (1, 2, 3) \succ (1, 2, 2) \\ &\succ (1, 2, 1) \succ (2, 2, 7) \succ (2, 2, 6) \succ (2, 2, 5) \succ (2, 2, 4) \\ &\succ (2, 2, 3) \succ (2, 2, 2) \succ (2, 2, 1). \end{aligned}$$

- For firm  $f_3$  we choose

$$Y_3^f := W \times \{f_3\} \times \{d_1, \dots, d_5\}$$

with quota  $q_3^f = 20$  and the following preference relation on  $Y_3^f$  (see (5.7)):

$$\begin{aligned} (4, 3, 1) &\succ (4, 3, 2) \succ (4, 3, 3) \succ (4, 3, 4) \succ (4, 3, 5) \\ &\succ (3, 3, 1) \succ (3, 3, 2) \succ (3, 3, 3) \succ (3, 3, 4) \succ (3, 3, 5) \\ &\succ (2, 3, 1) \succ (2, 3, 2) \succ (2, 3, 3) \succ (2, 3, 4) \succ (2, 3, 5) \\ &\succ (1, 3, 1) \succ (1, 3, 2) \succ (1, 3, 3) \succ (1, 3, 4) \succ (1, 3, 5). \end{aligned}$$

*Step 4.* Applying Theorem 3.13 we combine the three choice maps of the preceding step into a global revealing choice map  $C_F$  on  $W \times F \times D$  by setting

$$C_F(A) := \bigcup_{j=1}^3 C_{f_j}(A \cap (W \times \{f_j\} \times D)), \quad A \subset W \times F \times D.$$

*Step 5.* The choice maps  $C_W$  and  $C_F$  satisfy the hypotheses of Theorem 3.7. Applying the algorithm as described in Remark 3.8 (b) by starting with  $X_0 := X$ , we use a computer program to make the otherwise tedious computation. We obtain the following worker-optimal stable schedule:

$$(w_1, f_2, 1-4), \quad (w_2, f_1, 1-3), \quad (w_3, f_2, 2-3), \quad (w_4, f_2, 1-7).$$

The notations means that

- $f_1$  hires worker  $w_2$  for Mondays, Tuesdays and Wednesdays;
- $f_2$  hires worker  $w_1$  for Mondays, Tuesdays, Wednesdays and Thursdays, worker  $w_3$  for Tuesdays and Wednesdays, and worker  $w_4$  for all seven days of the week;

- $f_3$  does not hire anybody.

*Step 6.* Applying the algorithm of Remark 3.8 (b) by starting with  $Y_0 := X$  we obtain the same solution. This means that the worker-optimal and firm-optimal solutions coincide, and that there is a unique stable schedule matching in this case.

The remaining of this section investigates the changes in the solutions if we modify our requirements in various ways.

**5.2. Second problem.** If worker  $w_3$  accepts to work up to four days per week (so we change  $q_3^w = 2$  to  $q_3^w = 4$ ), then the worker-optimal and firm-optimal solutions still coincide: the stable schedule is given by the list

$$(w_1, f_2, 1 - 4), \quad (w_2, f_1, 1 - 3), \quad (w_3, f_2, 2 - 5), \quad (w_4, f_2, 1 - 7).$$

The only change with respect to the preceding case is that  $f_2$  now hires  $w_3$  for Thursdays and Fridays, too.

**5.3. Third problem.** We modify the problem such that  $f_2$  hires at most one worker per day, so that for the construction of the choice map  $C_{f_2}$  we add the extra conditions

$$Y_{1,k}^f := W \times \{f_2\} \times \{d_k\} \text{ and } q_{1,k}^f = 1 \text{ for } k = 1, \dots, 7.$$

The changes are more important. Both the worker-optimal solution and firm-optimal solutions are given by the list

$$(w_1, f_3, 1 - 4), \quad (w_2, f_1, 1 - 3), \quad (w_3, f_2, 2 - 3), \quad (w_4, f_2, 1, 4 - 7), \quad (w_4, f_3, 2 - 3).$$

**5.4. Fourth problem.** Now assume that

- firm  $f_1$  does not hire any worker for more than two days;
- firm  $f_2$  does not hire Worker  $w_1$  for more than three days;
- firm  $f_2$  does not hire Worker  $w_4$  for more than three days either.

We proceed as in Subsection 5.1 but in constructing  $C_1^f$  we add the extra conditions

$$\tilde{Y}_{i,1}^f := \{w_i\} \times \{f_1\} \times D \text{ and } \tilde{q}_{i,1}^f := 2 \text{ for } i = 1, 2, 3, 4,$$

and in constructing  $C_2^f$  we add the extra conditions

$$\begin{aligned} \tilde{Y}_{1,2}^f &:= \{w_1\} \times \{f_2\} \times D \text{ and } \tilde{q}_{1,2}^f := 3, \\ \tilde{Y}_{4,2}^f &:= \{w_4\} \times \{f_2\} \times D \text{ and } \tilde{q}_{4,2}^f := 3. \end{aligned}$$

Now the worker-optimal and firm-optimal stable schedules differ: they are given by

$$\begin{aligned} (w_1, f_2, 2 - 4), \quad (w_1, f_3, 1), \quad (w_2, f_1, 1 - 2), \quad (w_2, f_2, 3), \\ (w_3, f_2, 2 - 3), \quad (w_4, f_2, 5 - 7), \quad (w_4, f_3, 1 - 4) \end{aligned}$$

and

$$\begin{aligned} (w_1, f_2, 5 - 7), \quad (w_1, f_3, 1), \quad (w_2, f_1, 1 - 2), \quad (w_2, f_2, 3), \\ (w_3, f_2, 2 - 3), \quad (w_4, f_2, 5 - 7), \quad (w_4, f_3, 1 - 4), \end{aligned}$$

respectively.

## 6. CONCLUDING REMARKS

The *schedule matching* problem extends the standard matching procedure to the allocation of real numbers (days, hours or quantities) between two separate sets of agents. The present paper generalizes the notion of schedule matching to allow for schedule and preference constraints on each side of the market. We demonstrate, by several example, that the revealing property of the choice maps is the most suitable one to ensure the existence of stable matchings. We also revisit the mathematical structure of the matching theory by comparing various definitions of stable sets and various classes of choice maps.

The generality of our analysis is not only theoretically interesting but is potentially useful in application as well.

In certain, highly-competitive, labor markets employers perceive a shortage of top-level candidates that lead to hiring strategies intended to hire those who are believed to be the best. Competition within these paradigms inevitably leads to ever-evolving, if not escalating, dynamic reactions on both sides in an effort to maximize overall gain. As intended, the strategy forces the candidates make ever quicker decisions, before they can know, and weigh, other offers that may be proffered in the near future. As a consequence, candidates end up having less opportunities and employers less potential candidates than were originally available in the market. This results in sub-optimal matches that spawn a myriad of both observable and hidden costs on both sides of the market. The eventual failure of this common strategy ultimately mandates sets of new rules and procedures or market re-design. The algorithm proposed here could be used as a “clearinghouse” in situations where quotas are placed by workers on firms and days worked, allowing him to work part-time for different firms on the same day or on different days, excluding some firms on some given days or excluding some days of work. In the same manner, the algorithm is applicable to situations where firms need to adjust their labor force on certain days depending on their anticipated activity, or on the requirements associated to different activities on different days or the same day.

## 7. PROOF OF THE THEOREMS OF SECTION 3 AND SUPPLEMENTARY RESULTS

First we prove Propositions 3.5 and 3.6. Then we apply them to establish Theorem 3.7. In the second, independent part of the section we prove Theorems 3.10 and 3.13.

*Proof of Proposition 3.5.*

(a) Assume that  $C : 2^X \rightarrow 2^X$  is revealing. Then it is persistent because  $C(A) \subset A$  for every choice map.

In order to prove the consistence first we observe that in case  $C(A) \subset B$  we infer from our hypothesis  $C(A) \subset B \subset A$  and from the choice map property  $C(B) \subset B$  that  $C(B) \subset A$ . Therefore using (3.7) we have

and

$$C(B) \subset A \implies B \cap C(A) \subset C(B) \iff C(A) \subset C(B),$$

so that  $C(A) = C(B)$ .

Now assume that  $C : 2^X \rightarrow 2^X$  is consistent and persistent, and consider two sets satisfying  $C(A) \subset B$ . We have to prove that  $A \cap C(B) \subset C(A)$ .



Since  $A \subset A \cup B$ , applying (3.9) we obtain that

$$A \cap C(A \cup B) \subset C(A). \quad (7.1)$$

The proof will be completed by showing that  $C(A \cup B) = C(B)$ .

Using the hypothesis  $C(A) \subset B$  we deduce from (7.1) that  $C(A \cup B) \subset B$ . Therefore  $C(A \cup B) \subset B \subset A \cup B$ , and the equality  $C(A \cup B) = C(B)$  follows by applying (3.8).

(b) If the choice map is consistent, then its idempotence follows by applying (3.8) with  $B = C(A)$ . If  $C$  is persistent, then applying (3.9) with  $A = C(B)$  we get  $C(B) \subset C(C(B))$ . The converse inclusion also holds because  $C$  is a choice map.

(c) If  $A \subset B$ , then

$$\begin{aligned} R(A) \subset R(B) &\iff A \setminus C(A) \subset B \setminus C(B) \\ &\iff A \setminus C(A) \subset A \setminus C(B) \\ &\iff A \cap C(B) \subset C(A). \end{aligned} \quad \square$$

*Examples 7.1.*

(a) Consider a two-point set  $X = \{a, b\}$  and the four choice maps defined by the following formulae:

$A$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$C_1(A)$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a\}$
$C_2(A)$	$\emptyset$	$\emptyset$	$\{b\}$	$\emptyset$
$C_3(A)$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a, b\}$
$C_4(A)$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$

One may readily verify that

- $C_1$  is revealing,
- $C_2$  is persistent but not consistent,
- $C_3$  is consistent but not persistent,
- $C_4$  is not idempotent.

One may check that every idempotent choice map on  $X$  is either consistent or persistent (or both).

(b) Consider a three-point set  $X = \{a, b, c\}$  and the choice map  $C_5 : 2^X \rightarrow 2^X$  defined by

$$C_5(\{a\}) = \emptyset, \quad C_5(X) = \{b\}, \quad C_5(A) = A \quad \text{otherwise.}$$

Then  $C_5$  is idempotent but neither consistent, nor persistent.

*Proof of Proposition 3.6.*

(a) Using the idempotence of  $C_W$  and  $C_F$  we deduce from (3.12) that

$$S = C_F(S_F) = C_F(C_F(S)) = C_F(S).$$

(b) Assume that  $C_F$  is consistent (the other case is similar) set  $S'_F := S \cup (X \setminus S_W)$ . Then  $S_W \cup S'_F = X$ ,  $S_W \cap S'_F = S$  and we still have  $C_W(S_W) = S$ . Furthermore, since

$$C_F(S_F) = S \subset S'_F \subset S_F,$$

using the consistence of  $C_F$  we conclude that  $C_F(S'_F) = S$ .

(c) As we already observed (3.2)–(3.4) imply (3.12). Conversely, (3.12) contains (3.2); furthermore, by the consistency of  $C_W$  and  $C_F$ , (3.12) implies (3.3)–(3.4).

(d) If  $S$  is a stable set, then (3.5)–(3.6) follow from (3.4)–(3.7). Now assume (3.5) and (3.6). Setting

$$S_W := \{x \in X : C_W(S \cup \{x\}) = S\} \quad \text{and} \quad S_F := \{x \in X : C_F(S \cup \{x\}) = S\}$$

we have  $S_W \cup S_F = X$  by (3.6). In view of the consistence it remains to show that  $C_W(S_W) = S = C_F(S_F)$ .

If  $x \in S_W \setminus S$ , then applying the revealed preference property and using (3.6) we deduce from the inclusion  $C_W(S \cup \{x\}) \subset S_W$  that

$$(S \cup \{x\}) \cap C_W(S_W) \subset C_W(S \cup \{x\}) = S$$

and hence  $x \notin C_W(S_W)$ . We have thus  $C_W(S_W) \subset S$ . Applying again the revealed preference property we deduce from this last inclusion that

$$S_W \cap C_W(S) \subset C_W(S_W).$$

Since  $C_W(S) = S$  by (3.5), it follows that  $S \subset C_W(S_W)$ , so that finally  $C_W(S_W) = S$ . The proof of  $C_F(S_F) = S$  is similar.  $\square$

*Proof of Theorem 3.7.* Let us introduce the map  $f : 2^X \times 2^X \rightarrow 2^X \times 2^X$  by the formula

$$f(A, B) := (X \setminus R_F(B), X \setminus R_W(A))$$

where  $R_F, R_W$  denote the rejection maps corresponding to  $C_F$  and  $C_W$ . We observe that  $2^X \times 2^X$  is a non-empty complete lattice with respect to the order relation

$$(A, B) \leq (A', B') \iff A \subset A' \quad \text{and} \quad B \supset B'.$$

Furthermore,  $f$  is monotone with respect to this order relation. Indeed, using the monotonicity of the rejection maps we have

$$\begin{aligned} (A, B) \leq (A', B') &\iff A \subset A' \quad \text{and} \quad B \supset B' \\ &\implies R_W(A) \subset R_W(A') \quad \text{and} \quad R_F(B) \supset R_F(B') \\ &\implies X \setminus R_F(B) \subset X \setminus R_F(B') \quad \text{and} \quad X \setminus R_W(A) \supset X \setminus R_W(A') \\ &\iff f(A, B) \leq f(A', B'). \end{aligned}$$

Applying a fixed point theorem of Knaster and Tarski [10], [15], [16] we conclude that  $f$  has at least one fixed point and that the fixed points of  $f$  form a complete lattice. It remains to show that the fixed points of  $f$  coincide with the stable sets. More precisely, in view of Proposition 3.6 it is sufficient to prove that

$$f(A, B) = (A, B) \iff A \cup B = X \quad \text{and} \quad C_W(A) = A \cap B = C_F(B).$$

If  $f(A, B) = (A, B)$ , then  $A = X \setminus R_F(B)$  and  $B = X \setminus R_W(A)$ . Since  $R_F(B) \subset B$ , it follows from the first relation that  $A \cup B = X$ . Furthermore, the first relation also implies that  $A$  is the disjoint union of the sets  $X \setminus B$  and  $C_F(B)$  and hence that  $A \cap B \subset C_F(B) \subset A$ . Since  $C_F$  is a choice map, we also have  $C_F(B) \subset B$  and therefore  $C_F(B) = A \cap B$ . The proof of the equality  $C_W(A) = A \cap B$  is analogous.

Conversely, if  $A \cup B = X$  and  $C_W(A) = A \cap B = C_F(B)$ , then

$$\begin{aligned} X \setminus R_F(B) &= (X \setminus B) \cup C_F(B) \\ &= ((A \cup B) \setminus B) \cup (A \cap B) = (A \setminus B) \cup (A \cap B) = A \end{aligned}$$

and

$$\begin{aligned} X \setminus R_W(A) &= (X \setminus A) \cup C_W(A) \\ &= ((A \cup B) \setminus A) \cup (A \cap B) = (B \setminus A) \cup (A \cap B) = B, \end{aligned}$$

so that  $f(A, B) = (A, B)$ .  $\square$

*Remark 7.2.*

(a) In case  $X$  is a finite set, the proof of the theorem provides an efficient algorithm to find a stable set. Starting with  $(X_0, Y_0) := X \times \emptyset$  we define a sequence  $(X_1, Y_1), (X_2, Y_2), \dots$  by the recursive relations

$$(X_{n+1}, Y_{n+1}) := (X \setminus R_F(Y_n), X \setminus R_W(X_n)), \quad n = 0, 1, \dots,$$

i.e.,

$$X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n) \quad \text{and} \quad Y_{n+1} := (X \setminus X_n) \cup C_W(X_n), \quad n = 0, 1, \dots$$

Since we have obviously  $X_1 \subset X = X_0$  and  $Y_1 \supset \emptyset = Y_0$ , by the monotonicity of  $f$  we conclude that

$$X_0 \supset X_1 \supset \dots \quad \text{and} \quad Y_0 \subset Y_1 \subset \dots$$

Since  $X$  has only finitely many subsets, there exists an index  $n$  such that

$$(X_{n+1}, Y_{n+1}) = (X_n, Y_n),$$

and then  $S := X_n \cap Y_n$  is a stable set. As a matter of fact, we obtain in this way the worker-optimal stable set. Similarly, we may construct the firm-optimal stable set by the same recurrence relations if we start from  $(X_0, Y_0) := \emptyset \times X$ .

(b) If we start with  $(X_0, Y_0) := X \times \emptyset$ , then we obtain  $X_1 = X \setminus R_F(\emptyset) = X$  and therefore

$$X_0 = X_1, \quad Y_1 = Y_2, \quad X_2 = X_3, \quad Y_3 = Y_4, \dots \quad (7.2)$$

This implies that the above algorithm is equivalent to the more economical Gale–Shapley algorithm. There we start with  $X_0 := X$  and we compute successively

$$Y_1, X_2, Y_3, X_4, \dots$$

by using the recursive formulae

$$Y_{n+1} := (X \setminus X_n) \cup C_W(X_n) \quad \text{and} \quad X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n).$$

We stop when we obtain  $X_{n-1} = X_{n+1}$  for the first time, and we set  $S = C_W(X_{n-1})$ . Indeed, the equalities (7.2) and  $X_{n-1} = X_{n+1}$  imply that

$$X_{n-1} = X_n = X_{n+1} = X_{n+2} = \dots \quad \text{and} \quad Y_n = Y_{n+1} = Y_{n+2} = Y_{n+3} \dots$$

Therefore  $(X_{n+1}, Y_{n+1}) = (X_n, Y_n)$ , and

$$S = X_n \cap Y_n = X_{n-1} \cap ((X \setminus X_{n-1}) \cup C_W(X_{n-1})) = C_W(X_{n-1}).$$

In the last step we used that  $C_W(X_{n-1}) \subset X_{n-1}$  because  $C_W$  is a choice map.

Analogously, we may construct the firm-optimal stable set by starting with  $Y_0 := X$ , computing successively  $X_1, Y_2, X_3, Y_4, \dots$  by the same formulae as above, and setting  $S = C_F(Y_{n-1})$  for the first  $n$  such that  $Y_{n-1} = Y_{n+1}$ .

*Examples 7.3.*

(a) We cannot replace the revealed preference condition with the substitutes condition in Theorem 3.7. To show this consider the choice maps  $C_W := C_1$  and  $C_F := C_2$  of Example 7.1 (a) on the set  $X = \{a, b\}$ . Then  $C_W$  is revealing and  $C_F$  is persistent. However, there is no stable set. Indeed, we have  $C_W(S) = S = C_F(S)$  only if  $S = \emptyset$  or  $S = \{b\}$ , so that only these two sets are individually rational (see Remark 3.2). However,  $S = \emptyset$  is blocked by  $\{b\}$  because

$$C_W(S \cup \{b\}) = C_F(S \cup \{b\}) = \{b\} \neq S,$$

and  $S = \{b\}$  is blocked by  $\{a\}$  because

$$C_W(S \cup \{a\}) = \{a\} \neq S \quad \text{and} \quad C_F(S \cup \{a\}) = \emptyset \neq S.$$

Hence none of these sets is stable.

(b) We cannot replace the revealed preference condition with the consistence in Theorem 3.7 either. To show this consider the choice maps  $C_W := C_1$  and  $C_F := C_3$  of Example 7.1 (a) on the set  $X = \{a, b\}$ . Then  $C_W$  is revealing and  $C_F$  is consistent. However, there is no stable matching. Indeed, we have  $C_W(S) = S = C_F(S)$  only if  $S = \emptyset$ , so this is the only individually rational set. For  $S = \emptyset$  the condition (3.3) is satisfied only if  $S_W = \emptyset$ , and then  $S_F = X$  by (3.2). However, then  $C_F(S_F) = X \neq S$ , so that (3.4) fails.

Now we turn to the proofs of Theorems 3.10 and 3.13. They are independent of the preceding part of the present section.

*Proof of Theorem 3.10.* The choice map  $C$  remains the same if we change each  $Y_n$  to  $Y_n \cap Y$  in the construction. The choice map does not change either if we complete the family  $\{Y_n\}$  with  $Y' := Y \setminus \cup Y_n$  corresponding to the quota  $q' := \text{Card } Y'$ . Without loss of generality we assume henceforth that  $\{Y_n\}$  is a *partition* of  $Y$ , i.e.,  $Y$  is the *disjoint union* of the sets  $Y_n$ .

Let  $A, B \subset X$  be two sets satisfying  $C(A) \subset B$ ; we have to show that if  $y_k \in A \cap C(B)$  for some  $k$ , then  $y_k \in C(A)$ .

First we establish by induction on  $j$  the following inequalities:

$$\text{Card}(C_j(A) \cap Y_n) \leq \text{Card}(C_j(B) \cap Y_n) \text{ for all } n, \quad j = 0, \dots, k-1. \quad (7.3)$$

For  $j = 0$  our claim reduces to the trivial equality  $0 = 0$ . Assuming that the inequalities hold until some  $j < k-1$ , consider the (unique) index  $m$  for which  $y_{j+1} \in Y_m$ . For each  $n \neq m$  we have

$$C_j(A) \cap Y_n = C_{j+1}(A) \cap Y_n \text{ and } C_j(B) \cap Y_n = C_{j+1}(B) \cap Y_n$$

and therefore

$$\text{Card}(C_{j+1}(A) \cap Y_n) \leq \text{Card}(C_{j+1}(B) \cap Y_n)$$

by our induction hypothesis. For  $n = m$  the only critical case is when

$$y_{j+1} \in C_{j+1}(A) \setminus C_{j+1}(B).$$

Since  $y_{j+1} \in C(A)$  implies  $y_{j+1} \in B$  and since

$$\text{Card } C_j(B) \leq \text{Card } C_{k-1}(B) \leq q-1$$

because  $y_k \in C(B)$  and therefore

$$\text{Card } C_{k-1}(B) = \text{Card } C_k(B) - 1 \leq q-1,$$

by the construction this can only happen if

$$\text{Card}(C_j(A) \cap Y_m) \leq q_m - 1 \text{ and } \text{Card}(C_j(B) \cap Y_m) = q_m.$$

But then we have

$$\begin{aligned} \text{Card}(C_{j+1}(A) \cap Y_m) &= \text{Card}(C_j(A) \cap Y_m) + 1 \\ &\leq q_m \\ &= \text{Card}(C_j(B) \cap Y_m) \\ &= \text{Card}(C_{j+1}(B) \cap Y_m) \end{aligned}$$

as required.

Since  $y_k \in A \cap C(B)$ , we have  $y_k \in A$ . Furthermore, since  $C(A) \subset Y$  and the sets  $Y_n$  form a partition of  $Y$ , it follows from (7.3) that

$$\begin{aligned} \text{Card } C_{k-1}(A) &= \cup_n \text{Card}(C_{k-1}(A) \cap Y_n) \\ &\leq \cup_n \text{Card}(C_{k-1}(B) \cap Y_n) \\ &= \text{Card } C_{k-1}(B) \\ &= \text{Card } C_k(B) - 1 \\ &\leq q - 1 \end{aligned}$$

because  $C_k(B) \setminus C_{k-1}(B) = \{y_k\}$ .

Furthermore, in case  $y_k \in Y_n$  we have

$$(C_k(B) \cap Y_n) \setminus (C_{k-1}(B) \cap Y_n) = \{y_k\}$$

and therefore

$$\begin{aligned} \text{Card}(C_{k-1}(A) \cap Y_n) &\leq \text{Card}(C_{k-1}(B) \cap Y_n) \\ &= \text{Card}(C_k(B) \cap Y_n) - 1 \\ &\leq q_n - 1. \end{aligned}$$

Summarizing, the conditions (3.13)–(3.15) are satisfied and we conclude that  $y_k \in C(A)$  by construction. This completes the proof.  $\square$

*Remark 7.4.*

(a) The choice map constructed in Theorem 3.10 is consistent even if the sets  $Y_n \cap Y$  are not disjoint. Indeed, if  $C(A) \subset B \subset A$ , then comparing the construction of

$$C_0(A) \subset C_1(A) \subset \dots \text{ and } C_0(B) \subset C_1(B) \subset \dots,$$

we see that  $C_k(A) = C_k(B)$  for every  $k$  and therefore  $C(A) = C(B)$ . The equality  $C_k(A) = C_k(B)$  is obvious for  $k = 0$  because both sides are equal to zero. If it is true for some  $k - 1 \geq 0$ , then we have  $y_k \in C_k(A)$  if and only if

$$\begin{aligned} y_k &\in A, \\ \text{Card}(C_{k-1}(A) \cup \{y_k\}) &\leq q \\ \text{Card}((C_{k-1}(A) \cup \{y_k\}) \cap Y_n) &\leq q_n \text{ for all } n, \end{aligned}$$

and  $y_k \in C_k(B)$  if and only if

$$\begin{aligned} y_k &\in B, \\ \text{Card}(C_{k-1}(B) \cup \{y_k\}) &\leq q \\ \text{Card}((C_{k-1}(B) \cup \{y_k\}) \cap Y_n) &\leq q_n \text{ for all } n. \end{aligned}$$

Since  $C_{k-1}(A) = C_{k-1}(B)$  by the induction hypothesis, the equality  $C_k(A) = C_k(B)$  will follow if we show that  $y_k \in A \iff y_k \in B$  if the last two conditions are satisfied. Since  $C(A) \subset B \subset A$  and since  $C(B) \subset B$  ( $C$  is a choice map), we have

and

$$y_k \in B \implies y_k \in C(B) \implies y_k \in B \implies y_k \in A.$$

(b) The range of an idempotent choice map coincides with the set of its fixed points:

$$\{C(A) : A \subset X\} = \{A \subset X : C(A) = A\}.$$

*Example 7.5.* In Example 7.3 (b) the consistent choice map  $C_F$  cannot be obtained by the construction of Theorem 3.10 without the disjointness condition (see Remark 7.4 (a)). A stronger counterexample is the following. We consider a three-point set  $X = \{a, b, c\}$  and the following two choice maps:

$A$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
$C_W(A)$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{b, c\}$	$\{b, c\}$
$C_F(A)$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a\}$	$\{a, c\}$	$\{b\}$	$\{a, c\}$

Both choice maps are defined by the construction of Theorem 3.10. For  $C_W$  we take  $Y = X$  with the preference order  $c \succ b \succ a$  and quota  $q = 2$ , and we set  $Y_1 = \{a, c\}$  with the quota  $q_1 = 1$ . This is a revealing choice map. The choice map  $C_F$  is the one given in Example 3.12 above: a consistent but not revealing choice map because the disjointness condition is not satisfied.

In order to find a stable set  $S$  we have to cover  $X = \{a, b, c\}$  by two sets  $S_W$  and  $S_F$  satisfying  $C_W(S_W) = S = C_W(S)$  and  $C_F(S_F) = S = C_F(S)$ . The equalities  $C_W(S) = S = C_F(S)$  are satisfied if and only if  $S$  has at most one element, so that there are four candidates for the stable set  $S$ . We can see easily from the table that in order to have  $C_W(S_W) = S = C_W(S)$ ,

- in case  $S = \emptyset$  we must have  $S_W = S_F = \emptyset$ ;
- in case  $S = \{a\}$  we must have  $S_W = \{a\}$  and  $S_F \subset \{a, b\}$ ;
- in case  $S = \{b\}$  we must have  $S_W = \{b\}$  and  $S_F \subset \{b, c\}$ ;
- in case  $S = \{c\}$  we must have  $S_W \subset \{a, c\}$  and  $S_F = \{c\}$ .

Since  $S_W \cup S_F \neq X$  in all these cases, we conclude that there is no stable set.

*Proof of Theorem 3.13.* If  $C(A) \subset B$ , then setting  $A_i := A \cap X_i$  and  $B_i := B \cap X_i$  we have

$$\begin{aligned} A \cap C(B) \subset C(A) &\iff (A \cap C(B)) \cap X_i \subset C(A) \cap X_i \text{ for all } i \\ &\iff A_i \cap C_i(B_i) \subset C_i(A_i) \text{ for all } i. \end{aligned} \quad \square$$

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